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$SU(N)$ membrane $B \wedge F$ theory with dual-pairs

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Abstract

We construct a $SU(N)$ membrane $B \wedge F$ theory with dual pairs of scalar and tensor fields. The moduli space of the theory is precisely that of N M2-branes on the noncompact flat space. The theory possesses explicit $SO(8)$ invariance and is an extension of the maximal $SU(N)$ super-Yang-Mills theory.

Interestingly, recently certain type of matter Chern-Simons field theories in (1+2) dimensions have been proposed to be the low energy theories describing supermembranes. Amongst these, the originally proposed Bagger-Lambert-Gustavsson (BLG) theory has $\mathcal{N}=8$ $SO(8)$ superconformal invariance but the theory is known only for $SO(4)$ tri-algebra [1, 2]. Although for noncompact case of tri-algebras, BLG theory can be extended to admit $SU(N)$ symmetry [3]. But these theories have ghost fields in the spectrum and once these are gauge-fixed the theory eventually reduces to the $SU(N)$ super Yang-mills [4]. Another interesting class of matter-Chern-Simons theories proposed by Aharony-Bergman-Jafferis-Maldacena (ABJM) [5], however have ordinary Lie-algebra structure. These theories admit $\mathcal{N}=6$ $SU(N)_k \times SU(N)_{-k}$ superconformal symmetry, and is conjectured to be dual to M-theory on $AdS_4 \times S^7/Z_k$ with the level $k>2$. For $k=1,2$ the theory supposedly becomes maximally supersymmetric BLG theory.

It is now clear that the understanding of Chern-Simons theories is essential to know the M-theory origin of the $SU(N)$ Yang-Mills theory which describes N D2-branes on R^7, and vice-versa. In particular, in the works [6, 7] the authors have attempted to understand this link to some extent. The work [6] is of particular importance to us in this paper. We take a parallel but rather distinct approach where we augment the B-F theory with scalars and dual 2-rank tensor fields, $C^I_{(2)}$. This leads us to a membrane B-F theory which has $SU(N)$ gauge symmetry and has $SO(8)$ R-invariance as well as the scale invariance. The theory does not have any ghost degrees of freedom and also has no tri-algebras. It presumably also has maximal supersymmetry as it is simply the topological extension of the $3D$ super Yang-Mills theory.

The low energy $SU(N)$ super Yang-Mills theory with maximal supersymmetry is written as

$$\begin{aligned} S_{SYM} &= \int d^3x\ Tr(-\frac{1}{4g^2_{YM}}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu X^i D^\mu X^i - \frac{g^2_{YM}}{4}(X^{ij})^2 \\ &\quad + \frac{i}{2}\bar{\Psi}\gamma^\mu D_\mu \Psi + \frac{i}{2}\bar{\Psi}\Gamma_i[X^i, \Psi]) \end{aligned} \tag{1}$$

where we defined $X^{ij} = [X^i, X^j]$. The field strengths are

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]} - [A_\mu, A_\nu], \qquad D_\mu X^i = \partial_\mu X^i - [A_\mu, X^i] \tag{2}$$

The bosonic fields $(A_\mu; X^1, ..., X^7)$ are all in the adjoint of $SU(N)$ and the fermions Ψ^A_α form 2-compt spinor of $3D$ and 8-compt spinor of $SO(7)$. The theory has an explicit $SO(7)$ R-symmetry under which supercharges get rotated. The YM theory actually lives on the boundary of $AdS_4 \times S^6$ (with varying string coupling), which is the near horizon geometry of N D2-branes.

The scale (mass) dimensions are

$$[X^i] = \frac{1}{2},\ [A_\mu] = 1,\ [\Psi] = 1,\ [g_{YM}^2] = 1$$

Notice that the Yang-Mills coupling constant is dimensionful in $3D$! So the super Yang-Mills can hardly be a conformal theory. In fact the YM coupling has a flow. Although the theory has good high energy behaviour where it becomes a free theory in UV regime, but in IR it is known to flow to a strongly coupled superconformal fixed point. The conformal nature of the YM theory at the IR fixed point has remained illusive though. Whether it describes M2-brane theory has not been quite clear?

For several reasons it is expected that a theory of multiple M2-branes in flat space should have maximal supersymmetry, should be conformal, should have $SO(8)$ R-symmetry and possibly a gauge symmetry if it were an interacting theory. But the actual content of the theory has remained illusive so far. A way ahead was suggested by the authors [6] where one can make use of *de-Wit-Nicolai-Samtleben* duality transformations [8]. The dNS proposal is based on the fact that a propagating vector field in $3D$ contributes one degree of freedom. It is a familiar kind of Poincare duality between gauge field and a scalar field ($\frac{1}{2!}\epsilon^{\mu\nu\lambda}F_{\nu\lambda} = \partial^\mu\phi$). Instead in a non-Abelian situation we can define

$$\frac{1}{2!g_{YM}}\epsilon^{\mu\nu\lambda}F_{\nu\lambda} = D^\mu\phi - g_{YM}B^\mu \tag{3}$$

We can easily see that

$$-\frac{1}{4g_{YM}^2}Tr(F_{\mu\nu})^2 \equiv -\frac{1}{2}Tr(D^\mu\phi - g_{YM}B^\mu)^2 + Tr(\frac{1}{2}\epsilon^{\mu\nu\lambda}B_\mu F_{\nu\lambda}) \tag{4}$$

Thus the duality introduces a pair of adjoint fields B_μ and ϕ. This duality has been used in going from $SO(7)$ super-Yang-Mills to the BLG theory which has $SO(8)$ R-symmetry [6]. Actually after incorporating the dNS transformation the SYM Lagrangian takes the form of a matter B-F (BF) Lagrangian

$$\begin{aligned} S_{BF} = \int d^3x\, Tr(&-\frac{1}{2}(D^\mu\phi - g_{YM}B^\mu)^2 + \frac{1}{2}\epsilon^{\mu\nu\lambda}B_\mu F_{\nu\lambda} - \frac{1}{2}D_\mu X^i D^\mu X^i \\ &- \frac{g_{YM}^2}{4}(X^{ij})^2 + \frac{i}{2}\bar{\Psi}\gamma^\mu D_\mu\Psi + \frac{i}{2}\bar{\Psi}\Gamma_i[X^i,\Psi]) \end{aligned} \tag{5}$$

where we can now identify ϕ with X^8. This field alongwith the rest X^i's forms an $SO(8)$ vector: X^I $(1 \le I \le 8)$. One then also defines a coupling constant 8-vector: $g^I = (0,\cdots,0,g_{YM})$.

With this we can write BF Lagrangian in an $SO(8)$ covariant Lagrangian form [6]

$$S_{BF} = \int d^3x\, Tr(-\frac{1}{2}(D^\mu X^I - g^I B^\mu)^2 + \frac{1}{2}\epsilon^{\mu\nu\lambda} B_\mu F_{\nu\lambda} - U(g^I, X^I)$$
$$+ \frac{i}{2}\bar{\Psi}\gamma^\mu D_\mu \Psi + \frac{i}{2} g^I \bar{\Psi}\Gamma_{IJ}[X^J, \Psi]) \tag{6}$$

where the potential is defined as

$$U = \frac{1}{2.3!} V_{IJK} V^{IJK} \tag{7}$$

with the help of a completely antisymmetrized object

$$V_{IJK} = g_{[I} X_{JK]} \equiv g_I X_{JK} + \textit{cyclic permutations} \tag{8}$$

Specially, we must note that parameters g^I are in the 8_v while $X^{IJ} = [X^I, X^J]$ are in the adjoint of $SO(8)$ group. So as such the antisymmetrization of V_{IJK} should not be confused with any tri-algebra like in BLG theory. However, it can be extended to have a Lorentzian tri-algebra structure [6]. [1]

The action (6) has an $SO(8)$ invariance provided the couplings g^I transform along with various fields under $SO(8)$ rotations. Thus, although the theory has $SO(8)$ invariance but its action is transitive on the coupling parameters in the theory. After the transformations we get a new theory with a new set of couplings. The $\mathcal{N} = 8$ susy transformations can also be formally written in $SO(8)$ covariant form [6].

It is noteworthy here to mention that such phenomena have also been observed in the case of massive supergravity theory as well, see for an instance [9]. In the present scenario, the legitimate step would be like that in the Romans' theory in ten dimensions [10]. There we try to lift the mass parameter (cosmological constant) m to the level of a scalar field $M(x)$ which is then Hodge-dualised to a 10-form field strength F_{10} [11]. This does not introduce any new degree of freedom in the theory. Instead now the values of the mass parameters become localised in the spacetime. We shall like to implement the same idea here for the $3D$ case. Note that we have couplings g^I in the vector representation of $SO(8)$. So we first define correspondingly 8 scalar fields $\eta^I(x)$ such that

$$g^I = \ <\eta^I(x)>, \qquad g^I g^I = (g_{YM})^2 \tag{9}$$

In the next step, we introduce 2-form potential $C^I_{(2)}$, also in the 8_v, whose field strength will be dual to η^I. We must also make sure that the vacua are such that

[1] Here the $SO(8)$ gamma matrices are $\Gamma_8 = \tilde{\Gamma}^8$, $\Gamma^i = \tilde{\Gamma}^8\tilde{\Gamma}^i$. (The matrices with tilde will henceforth will be named as $SO(7)$ matrices.)

η^I will be constant. This can be done simply by introducing a new topological term in the $SO(8)$ covariant BF action

$$-\int C^I_{(2)} \wedge d\eta^I \tag{10}$$

which is $SO(8)$ invariant and has the gauge invariance under

$$C^I_{(2)} \to C^I_{(2)} + d\alpha^I_{(1)} \tag{11}$$

Thus the complete membrane action can be written as [2] [3]

$$\begin{aligned} S_{MBF} = \int d^3x\, Tr(-\frac{1}{2}(D^\mu X^I - \eta^I B^\mu)^2 + \frac{1}{2}\epsilon^{\mu\nu\lambda} B_\mu F_{\nu\lambda} - \frac{1}{2.3!}(V_{IJK})^2) \\ -\frac{1}{2}\epsilon^{\mu\nu\lambda} C^I_{\mu\nu}\partial_\lambda \eta^I + S_{fermions} \end{aligned} \tag{12}$$

where

$$D^\mu X^I = \partial^\mu X^I - [A_\mu, X^I], \qquad V_{IJK} = \eta_{[I} X_{JK]} \ .$$

The equations of motion are now augmented with two new set of equations. Namely the C^I equation

$$\partial_\lambda \eta^I = 0 \tag{13}$$

and the η^I equation

$$Tr((D^\mu X^I - \eta^I B^\mu)B_\mu - \frac{1}{2}V^{IJK}X_{JK}) + \frac{1}{2}\epsilon^{\mu\nu\lambda}\partial_\mu C^I_{\nu\lambda} = 0 \tag{14}$$

The C^I-equation implies that all η^I are constant. The second equation only relates η^I with its dual tensor field $C^I_{\nu\lambda}$ and should be taken as the duality relation. The rest of the field equations remain unchanged. So the net content of the theory remains intact. There are no free parameters in the theory. The action (12) also has scale invariance. The supersymmetry presumably can also be made manifest which we do not work out here. Henceforth we shall refer to the action (12) as membrane B-F (MBF) theory.

Thus in bringing the BF theory to the MBF form we have actually introduced dual pairs of fields (C^I, η^I). The introduction of these dual pairs has introduced a new paradigm in the MBF theory. The moduli space of vacua in the MBF theory

[2] At this point we may be tempted to add another possible topological term $-\theta \int C_{(3)}$, as it does not affect any of the dynamical considerations. Although from topological perspectives it will be necessary.

[3] The $C_{(3)}$ can also be relevant while quantising the theory in the nontrivial membrane background. I thank S. Mukhi for this useful remark.

is now larger than the original SYM/BF theory. To know the moduli space of the MBF theory we need to solve

$$\begin{aligned} &\partial_{\eta^I} U(\eta, X) - \frac{1}{2!}\epsilon^{\mu\nu\lambda}\partial_\mu C^I_{\nu\lambda} = 0 \\ &\quad \to \eta^{[I} Tr(X^{JK]} X_{JK}) - \epsilon^{\mu\nu\lambda}\partial_\mu C^I_{\nu\lambda} = 0 \end{aligned} \tag{15}$$

and

$$\partial_{X^I} U(\eta, X) = 0 \tag{16}$$

These equations have quite a few interesting possibilities.

Case-1: We take first $C^I_{\mu\nu} = constt$. Since the solution of $\eta^I(x) = g^I$, we find that we need to have

$$X^{IJ} = [X^I, X^J] = 0. \tag{17}$$

This can happen when all the X^I's are taken to be diagonal matrices. That means all M2-branes are coincident. Hence the moduli space is exactly that of N coincident M2-branes on noncompact R^8 space.

However, the special case can arise when we take

$$\eta^8 = g_{YM},\ \eta^i = 0\ . \tag{18}$$

This will then require

$$X^{ij} = 0\ . \tag{19}$$

In the simplest case all X^i can be taken diagonal, but matrices X^8 can still be nontrivial but constant. These presumably will be the desired Goldstone modes corresponding to the spontaneously broken $SO(8)$ invariance. These will be eaten up by B_μ fields and making them heavy which can be integrated out in order to make the A_μ fields dynamical. All this precisely corresponds to the moduli space of N D2-branes on R^7.

For both of the above solutions the components V_{IJK} are vanishing hence the scalar potential altogether vanishes. So these would make the maximally supersymmetric vacua in MBF theory.

Case-2: Another rather interesting case is of 3D domain-walls. Let us take the tensor components C^I_{01} to be linearly dependent on one of the space coordinates, x_2 (say), then

$$dC^I \sim m^I dx^0 \wedge dx^1 \wedge dx^2 \tag{20}$$

is nontrivial, the m^I being the slope parameters. The two such phases with different slopes can be separated via domain-walls which are just the line defects in 2-dimensional plane. In this situation, we shall have g^I and m^I related via

$$\frac{1}{2} g^{[I} Tr(X^{JK]} X_{JK}) - m^I = 0 \tag{21}$$

This will describe a noncommuting (*fuzzy*) configuration of membranes. However, we are not sure if any static nontrivial fuzzy configuration can be found in which eq. (16) will be simultaneously satisfied. In any case, it will be interesting to find nonstatic solutions.

Quantisation:

At this point, let us also discuss an interesting quantum aspect which follows straightforwardly from action (12). The equation of motion for X^I,

$$\frac{1}{2!}\epsilon^{\mu\nu\lambda}F_{\nu\lambda} = (D^\mu X^I - \eta^I B^\mu)\eta^I \ , \tag{22}$$

and the equation (14) can be combined to give

$$Tr(\frac{1}{2!}\epsilon^{\mu\nu\lambda}F_{\nu\lambda}B_\mu - U) + \frac{1}{2}\eta^I\epsilon^{\mu\nu\lambda}\partial_\mu C^I_{\nu\lambda} = 0 \tag{23}$$

In the vacuum where $U = 0$, it has interesting implications. For example, consider an Euclidean monopole configuration where $F_{\mu\nu} \neq 0$ inside a 3-dimensional volume V^3, with a boundary $\partial V^3 \sim S^2$. We can have a configuration where

$$Tr\frac{1}{4\pi}\int_{V^3} B \wedge F \sim p(N) \ , \tag{24}$$

Here we have taken $p(N) \in \mathbf{Z}$ to depend upon the rank N of the Yang-Mills group. The actual expression of $p(N)$ however will depend upon the details of the monopole configuration. We are taking $SO(7)$ configuration where $\eta^8 = g_{YM}, \ \eta^i = 0$. The equation (23) leads us to the quantization

$$-\frac{1}{4\pi\sqrt{l_p}}\int_{V^3} dC_{(2)} = -\frac{1}{4\pi\sqrt{l_p}}\int_{S^2} C_{(2)} = k \in \mathbf{Z} \ . \tag{25}$$

with $g_{YM} \sim \frac{p(N)}{(l_p)^{1/2}k}$, and we introduced l_p, the 11-dimensional Planck length. That is we need to have a nontrivial $C_{(2)}$ flux over S^2. It does mean that Yang-Mills coupling in a given topological vacuum is controlled by the ratio of $p(N)$ and k. By having large k limit we can accomodate a weak Yang-Mills coupling. This argument appears almost analogous to large k limit in C_4/Z_k orbifold models [5].

In summary, we have taken an approach where we augmented the B-F theory with scalars and dual 2-rank tensor fields, namely $(\eta^I, \ C^I_{(2)})$. This led us to a membrane B-F theory which has $SU(N)$ gauge symmetry and has $SO(8)$ R-invariance as well as the scale invariance. There are no free parameters in the action. The theory does not have any ghost degrees of freedom and also has no tri-algebras. So in that aspect our theory is distinct from the B-F theory of [6]. Our theory presumably also has maximal supersymmetry as it is simply the topological extension of the $3D$

super Yang-Mills theory. Interestingly, the moduli space comes out to be that of N coincident M2-branes on transverse R^8.

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